Finite reflection groups and Coxeter groups

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Overview

The definition of reflection

Let V be a euclidean space with bilinear form (α, β).

Definition

A reflection s_α is a nonidentical motion of the euclidean space, whose set of the fixed points is the hyperplane H_α orthogonal to vector α.

It is clear, that

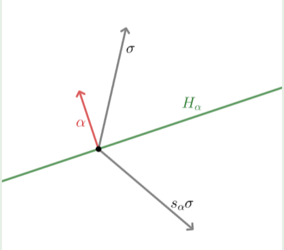
$$s_\alpha \sigma = \sigma - \frac{2(\alpha, \sigma)}{(\alpha, \alpha)} \alpha.$$

Trivial properties of reflections

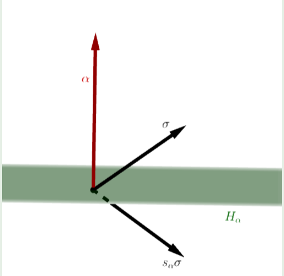
Properties

1. If $v \in H_\alpha$, then $s_\alpha v = v$.
2. $(s_\alpha \sigma, s_\alpha \mu) = (\sigma, \mu)$, i.e. s_α is an orthogonal transformation.
3. $s_\alpha^2 = 1$, i.e. reflection has order 2 in $O(V)$.

Examples



2d



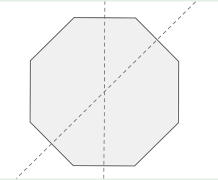
3d

Finite reflection groups

The subject of our further study is a finite group W generated by reflections. The main goal is to classify and describe all of them.

Examples

- Dihedral group. All elements of this group, which are, in fact, rotations around the center and "equatorial" symmetries, can be obtained by composition of two neighboring symmetries, like on the picture.
- Symmetric group. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a basis of n-dim space, with symmetric group S_n acting by permutations of this basis. Here reflections are s_α, $\alpha = \varepsilon_i - \varepsilon_j$.



Properties of finite reflection groups

From now on denote by W a finite reflection group. Each vector α decomposes V into a direct sum of hyperplane H_α orthogonal to it and a line $L_\alpha = \mathbb{R}\alpha$, and, indeed, W permutes such lines L_α, with $s_\alpha \in W$. This follows from the following fact:

Proposition

$ts_\alpha t^{-1} = s_{t\alpha},\ t \in O(V),\ \alpha \in V\backslash\{0\}.$

Hence, for $s_\alpha, w \in W$ the reflection $s_{w\alpha}$ is also in W, and it yields $w(L_\alpha) = L_{w\alpha}$. If we select unit vectors lying in all such lines, the so obtained collection will be stable under the action of W – and this suggests to distinguish such special systems of vectors.

Root system

Definition

A finite set Φ of nonzero vectors of V is called a root system if

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}, \ \forall \alpha \in \Phi$.
2. $s_\alpha \Phi = \Phi, \ \forall \alpha \in \Phi$;

Vectors $\alpha, \ \alpha \in \Phi$ are called roots.

It turns out that, marvelously, root systems are a very convenient tool for studying reflection groups. It is clear, that Φ defines W and vice versa.

Theorem

A group generated by reflections s_α for α roots of a root system, is finite.

Redefine W as a group generated by some root system Φ. The next goal is to study such systems.

Positive systems

Root system Φ may be extremely large. For example, when W is a dihedral group, Φ may have just as many elements as W. This leads us to look for a linearly independent subset of Φ called a simple system. Let us start by totally ordering V and defining a positive system.

Recall

Total ordering of real vector space V is a transitive relation on V, satisfying

1. $\forall \lambda, \mu \in V$, exactly one of $\lambda < \mu, \lambda = \mu, \mu < \lambda$ holds.
2. $\forall \gamma, \mu, \nu \in V$, if $\mu < \nu$, then $\lambda + \mu < \lambda + \nu$.
3. If $\mu < \nu$ and c is nonzero real number, then $c\mu < c\nu$ if $c > 0$, while $c\nu < c\mu$ if $c < 0$.

Positive and simple root systems

Definition

We say that $\lambda \in V$ is positive if $0 < \lambda$.
Π is a positive system if it consists of all positive roots relative to some total ordering of V.

Definition

Subset Δ of Φ is a simple system (and its elements are simple roots) if

1. Δ is a vector space basis for the $\mathbb{R}$-span of Φ in V;
2. each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.

Theorem on simple systems

Theorem

1. *If Δ is a simple system in Φ, then there is a unique positive system containing Δ.*
2. *Every positive system $\Pi \subset \Phi$ contains a unique simple system.*
3. *Simple system exists.*

The proof of this theorem uses the following important geometric fact:

Corallary

If Δ is a simple system in Φ, then $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in Δ.

Isomorphisms

Theorem

Any two simple systems in Φ are conjugate under W

Definition

$m(\alpha, \beta)$ denotes the order of $s_\alpha s_\beta$ in W for any roots α, β.

Theorem

Fix a simple system $\Delta \in \Phi$. Then W is generated by the set $S := \{s_\alpha, \alpha \in \Delta\}$, subject only to the relations:

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1(\alpha, \beta \in \Delta).$$

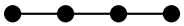
In other words, W is determined up to isomorphism by the set of integers $m(\alpha, \beta), \alpha, \beta \in \Delta$.

Coxeter graphs

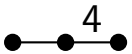
A convenient way to encode this information in a picture is to construct a graph Γ with a vertex set in one-to-one correspondence with Δ; join a pair of vertices corresponding to $\alpha \neq \beta$ by an edge whenever $m(\alpha, \beta) \geq 3$, and label such an edge with $m(\alpha, \beta)$. This graph is called **Coxeter graph**, and it determines W up to isomorphism.

Here are some examples of Coxeter graphs:

A_4 :



C_3 :



Matrices associated with Coxeter graphs

Definition

Let us associate to a Coxeter graph Γ with vertex set S of cardinality n a symmetric $n \times n$ matrix A by setting

$$a(\alpha, \beta) := -cos \frac{\pi}{m(\alpha, \beta)}$$

It is clear, that such an element is non-positive when $\alpha \neq \beta$
Here is an example of such a matrix for A_4:

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

Graphs of positive type

Definition

Matrix A is called *positive definite* if $x^T Ax > 0$ for all $x \neq 0$, and *positive semidefinite* if $x^T Ax \geq 0$ for all x.
We call a Coxeter graph positive definite (semidefinite) if it's adjacency matrix is positive definite (semidefinite).

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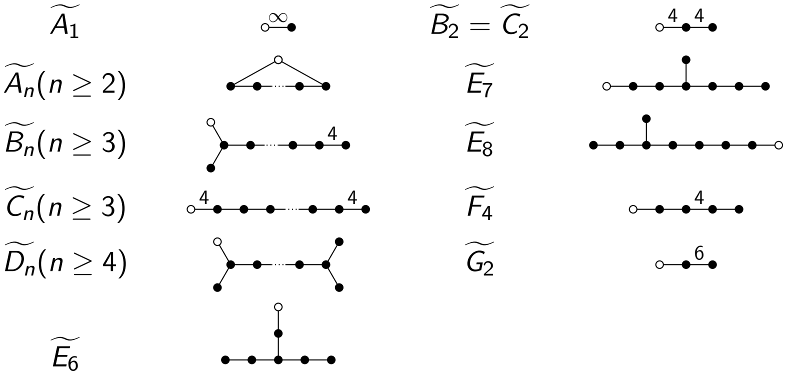
- Notice that Coxeter graph is positive definite since it represents the Gramm matrix of the corresponding bilinear form in the euclidean space.
- There is another characterization of positive type matrices. The matrix A is positive definite (semidefinite) if all its principal minors are positive (nonnegative).

Some positive semidefinite graphs

- As a tool in the proof of the main classification, we list some positive semidefinite (but not definite) graphs:



Subgraphs

Proposition

Let A be a real symmetric $n \times n$ matrix which is semidefinite and indecomposable. Assume, that $a_{ij} \leq 0$ whenever $i \neq j$. Then:

1. $N := \{x \in \mathbb{R}^n | x^t Ax = 0\}$ *coincides with the null space of A and has dimension ≤ 1.*
2. *The smallest eigenvalue of A has multiplicity 1 and has eigenvector whose coordinates are all positive.*

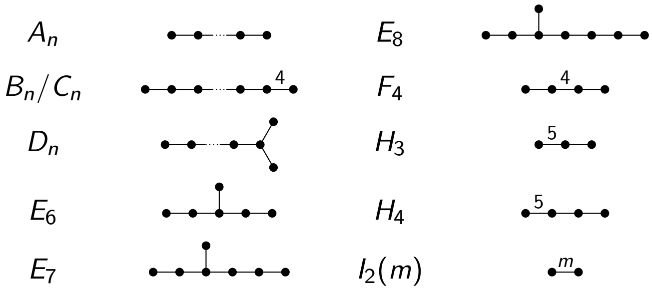
Corollary

If Γ is a connected Coxeter graph of positive type, then every proper subgraph is positive definite.

Classification of graphs of positive type

Theorem

The graphs in the figure below are the only connected Coxeter graphs of positive type.



Crystallographic groups

Definition

Subgroup G of $GL(V)$ is said to be *crystallographic* if it stabilizes the lattice L in V (the $\mathbb{Z}$-span of basis of V): $gL \subset L$ for all $g \in G$.

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Proposition

If W is crystallographic, then each integer $m(\alpha, \beta)$ must be either $2, 3, 4$ *or* 6 *for $\alpha \neq \beta$ in Δ.*

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- This proposition rules out groups of type H_3, H_4 as well as dihedral groups except those of order $2, 4, 6, 8$ and 12.

Crystallographic root systems

Definition

The root system Φ is called *crystallographic* if it satisfies an additional requirement:

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$

(Actually it is enough to require these ratios to be integers for $\alpha, \beta \in \Delta$).

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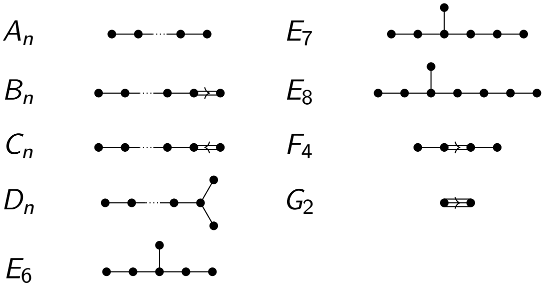
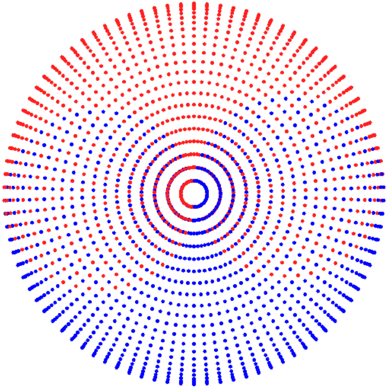
- Group generated by reflections s_α where $\alpha \in \Phi$ is *crystallographic*.

Dynkin diagrams

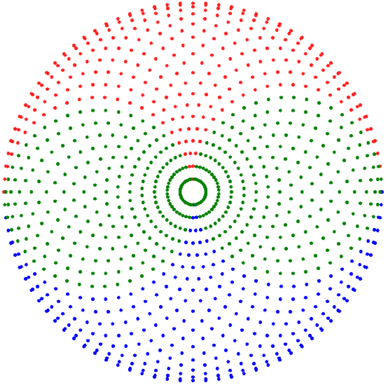
- It can be shown that if W is irreducible, then at most two root lengths are possible: 'short' and 'long' roots. This information is added to the Coxeter graph by directing an arrow on the edge towards the short root.
- Labels 4 and 6 are replaced with double and triple edges respectively.



Coxeter plane

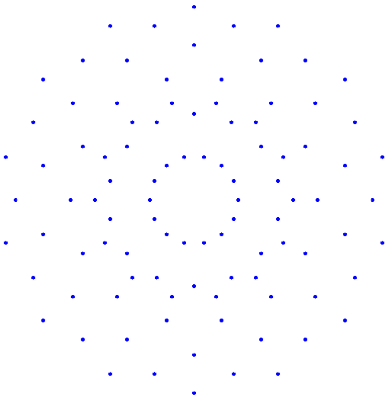


A_{45}

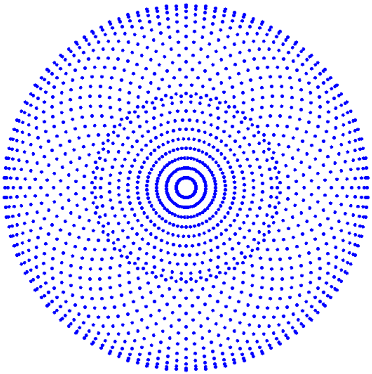


A_{46}

Coxeter plane

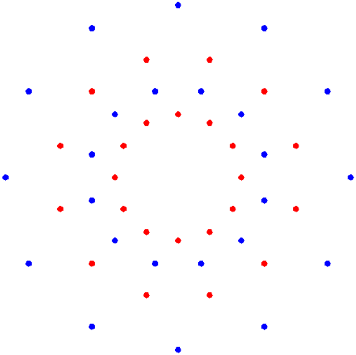


D_8



D_{30}

Coxeter plane



F_4